Lecture 1: Intro and Difference Equations

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Lecture Objectives:

- Introduce the course.
- Explain Difference Equations solutions and their properties.
- The iteration method for solving DE.
- Present the general solution to a DE by solving the homogeneous DE and finding the particular solution.
- Introduce two methods for finding a particular solution: method of undetermined coefficients and lag operators.

Main reading:

- Lecture notes

Secondary Readings:

- Applied Econometric Time Series, Enders, Walter, Fourth Edition
- Time Series Analysis, Hamilton, James, first edition
- Time Series Analysis and Its Applications: with R examples, Shumway, Robert H., Stoffer, David S., 2011

Intro to R

- Little Book of R for Time Series!

Macroeconometrics: Very large and dynamic field

Macroeconometrics deals with the empirical analysis of macro data and theory.

- In this course we focus on the large subset of macroeconometrics dealing with time series.
- Time series can serve a dual-purpose: (1) understanding better dynamic models; (2) forecast.

Macroeconometrics: A brief history

- Jan Tinbergen
  1. Was one of the founders of the field which is now called Macroeconometrics
  2. Treated economies as a system of equations
  3. Estimated parameters of these equations

- The Cowles Commission approach (30s-70s)
  1. Its main agenda was to determine the effect of exogenous (policy) variables on macroeconomic indicators
  2. They advocated an empirical method based on three steps: Specification, Estimation and Simulation

- The Cowles Commission fails in the 70s
  1. Structural changes
  2. Lucas Critique

Macroeconometrics: A brief history

- New Approach: VAR and SVAR
  1. Chris Sims introduces VAR as a parsimonious way to describe the dynamics properties of the macro data
  2. SVAR focuses on imposing restrictions to make the model reflect theory
  3. The focus of the VAR approach is shocks. The prime step is to define the appropriate shocks and then to analyze the response of the system to shocks by looking at impulse responses and variance decompositions

- Dynamic Stochastic General Equilibrium (DSGE)
  1. Based on microfoundations, it is able to overcome the Lucas Critique. Allows for policy evaluation!
  2. At the same time, it provides a strong narrative and interpretation in contrast to VAR.
  3. Finally, its forecast capabilities are reaching an accuracy level close to VAR.

Course Road Map

- Basic concepts in time-series analysis - Lectures 1, 2 and 3.
- VAR models - Lecture 4.
- Modeling trends and cycles - Lectures 5 and 6.
- Bayesian Analysis - Lecture 7.
- Models for unobserved components - Lectures 8 and 9.
- Modeling volatility and correlation - Lecture 10.
- Non-linear models - Lecture 11.

- Midterm and Final Exam - Check Syllabus.

Time Series: Example 1

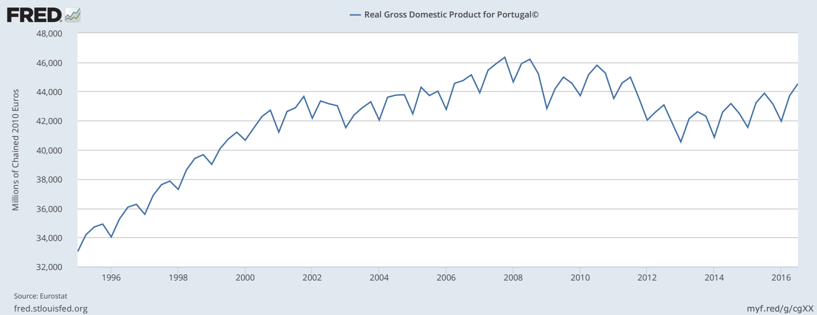
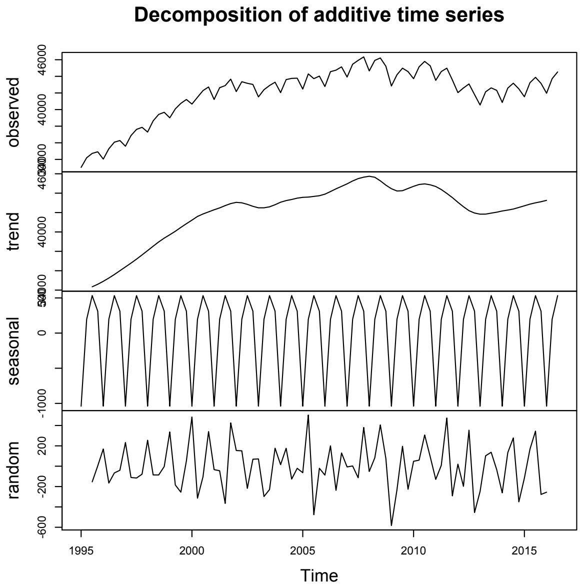


Figure: Portugal real GDP - Not Seasonally Adjusted

Time Series: Decomposition



Time Series: Example 2

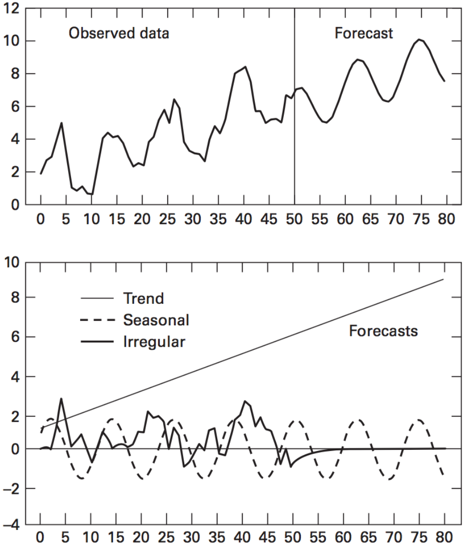


Figure: Hypothetical Time Series - Enders

Time Series: Example 2

- The series can again be decomposed into 3 components: Trend, Seasonal and Irregular. Lets formalize each of them mathematically:

$$y_t = T_t + S_t + I_t \tag{1}$$

$$\begin{aligned} T_t &= 1 + 0.1t \qquad (2) \\ S_t &= 1.6\sin(t\pi/6) \qquad (3) \\ I_t &= 0.7I_{t-1} + \varepsilon_t \qquad (4) \end{aligned}$$

  where T_t is the value of Trend at time t, S_t the Seasonal, and I_t the Irregular.

- Each of them is a type of difference equation.

Difference Equations

- In its most general form, a difference equation expresses the value of a variable as a function of its own lagged values, time, and other variables.
- Time-series econometrics is concerned with the estimation of difference equations containing stochastic components.
- The time-series econometrician may estimate the properties of a single series or a vector containing many interdependent series (next topic in the course).

Difference Equations

- The first difference up to the nth-difference are defined as: $\Delta y_t = y_t - y_{t-1}, \Delta y_{t+1} = y_{t+1} - y_t, \ldots, \Delta y_{t+n} = y_{t+n} - y_{t+(n-1)}$

  Example 1:

$$y_t = y_{t-1} + 2 \Rightarrow$$

$$\Delta y_t = 2 \tag{5}$$

- A **solution** to the first order difference equation above is a function $y_t(t, y_0, x_t)$! (not a number)

Difference Equations

- The solution is given by:

$$y_t = 2t + C \tag{6}$$

Lets verify (6) is indeed a solution to (5).

Example 2:

$$I_t = 0.7I_{t-1} + \varepsilon_t \tag{7}$$

- Its **solution** is given by:

$$I_t = \sum_{i=0}^{\infty} (0.7)^i \varepsilon_{t-i} \tag{8}$$

Verify (8) is a solution to (7).

Iteration Method

- So far, the solution was given to you. Here, we show how to use the iteration method in order to find a solution.
- Lets start with the simple difference equation example:

$$y_t = a_1 y_{t-1} + \varepsilon_t \tag{9}$$

  where a_1 is a constant coefficient and ε_t is an exogenous shock.
- If an initial condition is given to us y_0, we can use it and iterate forward to get a solution to the difference equation.

Iteration Method

- After iteration we get:

$$y_t = a_1^t y_0 + \sum_{i=0}^{t-1} a_1^i \varepsilon_{t-i} \tag{10}$$

- Hence we get a solution $y_t(t, y_0, x_t)$, where $x_t = \sum_{i=1}^{t-1} a_1^i \varepsilon_{t-i}$

  Exercise: Show (10) is a solution to (9). Exercise: Start with y_t and iterate backwards. Show the result is the same as in (10).

Iteration Method

- Suppose the initial condition is not known.

$$y_t = a_1^t y_0 + \sum_{i=0}^{t-1} a_1^i \varepsilon_{t-i} \tag{11}$$

Then y_0 is still unknown. We can continue to iterate back. Lets iterate m times back and we get:

$$y_t = a_1^{t+m} y_0 + \sum_{i=0}^{t+m-1} a_1^i \varepsilon_{t-i} \tag{12}$$

- If $|a_1| > 1$, then there is nothing we can do.

Iteration Method

- However, if $|a_1| < 1$ then as m becomes large, the first term of (12) goes to zero.
- If $|a_1| < 1 \Rightarrow \lim_{m \to \infty} a_1^{t+m} = 0$. Hence:

$$y_t = \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i} \tag{13}$$

is a solution to (9). Check this is the case.

Iteration Method

- The solution is not unique. For any given A,

$$y_t = Aa_1^t + \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i} \tag{14}$$

is also a solution to (9). Check again this is the case.

- Again, the constant A can only be pin downed with an initial condition. Given y_0 we can find A:

$$A = y_0 - \sum_{i=0}^{\infty} a_1^i \varepsilon_{0-i} \tag{15}$$

Random Walk

- Suppose $a_1 = 1$. Then, we have what is called a random walk process. Example: Stock prices under the efficient market hypothesis.

$$y_t = y_{t-1} + \varepsilon_t \tag{16}$$

  The solution (check!) is given by:

$$y_t = \sum_{i=1}^{t} \varepsilon_i + y_0 \tag{17}$$

- Thus, in this process, the exogenous shocks have a permanent non-decaying effect on y_t. Its also called long memory model.

Examples with different coefficients

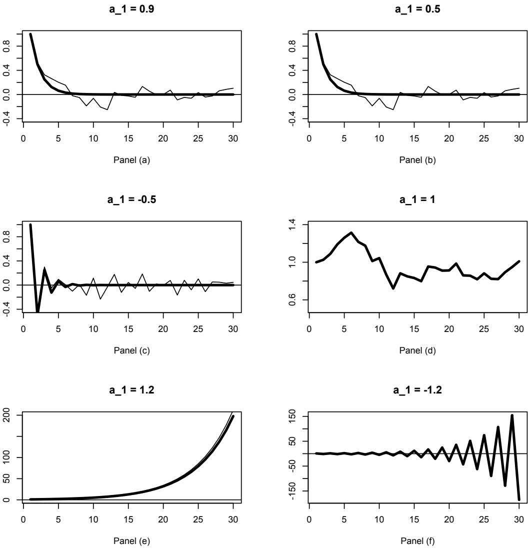


Figure: Walter Enders Figure 1.2 Replication

General Solution

- The Iteration Method becomes unfeasibly at higher orders of difference equation.
- To illustrate the general method of finding a solution, lets consider the same difference equation:

$$y_t = a_1 y_{t-1} + \varepsilon_t \tag{18}$$

- Lets focus on the homogeneous part of (18).

$$y_t = a_1 y_{t-1} \tag{19}$$

The solution to this homogeneous equation is called the **homogeneous solution**. In this case:

$$y_t^h = A a_1^t \tag{20}$$

General Solution

- With the aid of the thick lines in Figure 1.2, we can classify the properties of the homogeneous solution as follows:
    1. If $|a_1| < 1$, the homogeneous solution converges.
    2. If $|a_1| > 1$, the homogeneous solution explodes.
    3. If $a_1 = 1$, the homogeneous solution is constant and equal to A. If $a_1 = -1$, oscillates with the same frequency and in the same interval between A and $-A$.
- Now consider (18) again. We found that (13) was a solution to (18). This is called the **particular solution** of (18). It is particular because it is not unique.

General Solution

- The homogeneous solution (20) plus the particular solution given by (13) constituted the complete solution to (18).
- The general solution to a difference equation is defined to be a particular solution plus all homogeneous solutions.
- Once the general solution is obtained, the arbitrary constant A can be eliminated by imposing an initial condition for y_0.

General Solution

- The results can be extended for the nth-order difference equation
- In this general case, it will be more difficult to find the particular solution and there will be n distinct homogeneous solutions.

  The general steps are the following:

  1. form the homogeneous equation and find all n homogeneous solutions;
  2. find a particular solution;
  3. obtain the general solution as the sum of the particular solution and a linear combination of all homogeneous solutions;
  4. eliminate the arbitrary constant(s) by imposing the initial condition(s) on the general solution.

Solving Homogeneous Difference Equations

- Lets start with a second-order homogeneous difference equation:

$$y_t = a_1 y_{t-1} + a_2 y_{t-2}$$

$$y_t - a_1 y_{t-1} - a_2 y_{t-2} = 0 \tag{21}$$

- Lets use our guess solution from the first-order case, $y_t^h = A\alpha^t$, and substitute it into (21):

$$A\alpha^t - a_1 A\alpha^{t-1} - a_2 A\alpha^{t-2} = 0 \tag{22}$$

  Divide both sides by $A\alpha^{t-2}$ and we get the characteristic equation:

$$\alpha^2 - a_1\alpha - a_2 = 0 \tag{23}$$

Solving Homogeneous Difference Equations

- Solving the quadratic equation gives the two characteristic roots:

$$(\alpha_1, \alpha_2) = \left(\frac{a_1 - \sqrt{a_1^2 + 4a_2})}{2}, \frac{a_1 + \sqrt{a_1^2 + 4a_2})}{2} \right)$$

- Again, these solutions are not unique. In fact, for any two arbitrary constants A_1 and A_2, the linear combination $A_1(\alpha_1)^t + A_2(\alpha_2)^t$ also solves (21). Check!

- The complete homogeneous solutions is given by:

$$\boxed{y_t^h = A_1(\alpha_1)^t + A_2(\alpha_2)^t}$$

Stability Conditions

Proposition: *If the characteristic roots lie within the unit circle, the homogeneous solution is convergent.*

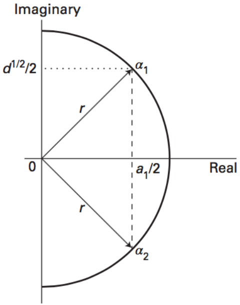


Figure: Unit Circle Example

Finding the Particular Solution

- There are many methods that can help in finding a particular solution to non-homogeneous difference equations.

  1. **Method of Undetermined Coefficients**
  2. **Lag operators**
  3. Diagonalization (Particularly useful in DSGEs)
  4. Variation in Parameters

Method of Undetermined Coefficients

- This method is particularly useful in linear systems.
- The main idea comes form the fact that linear difference equations generally have linear solutions.

  For instance, we found

  $$y_t = a_1 y_{t-1} + \varepsilon_t \tag{24}$$

  had a linear particular solution:

  $$y_t = \sum_{i=0}^{\infty} a^i \varepsilon_{t-i} \tag{25}$$

- The method relies on posing a challenge solution and then substituting this challenge solution into the difference equation. Finally we match the coefficients. If we are able to match it we have found the solution.

Method of Undetermined Coefficients

- Suppose we are again facing:

$$y_t = a_1 y_{t-1} + \varepsilon_t \tag{26}$$

- Our challenge solution is the following:

$$y_t = b_1 t + \sum_{i=0}^{\infty} \alpha_i \varepsilon_{t-i} \tag{27}$$

If we substitute our challenge solution in (26) we have:

$$b_1 t + \alpha_0 \varepsilon_t + \alpha_1 \varepsilon_{t-1} + \ldots = a_1 (b_1 (t-1) + \alpha_0 \varepsilon_{t-1} + \alpha_1 \varepsilon_{t-2} + \ldots) + \varepsilon_t$$

Method of Undetermined Coefficients

$$b_1 t + \alpha_0 \varepsilon_t + \alpha_1 \varepsilon_{t-1} + ... = a_1(b_1(t-1) + \alpha_0 \varepsilon_{t-1} + \alpha_1 \varepsilon_{t-2} + ...) + \varepsilon_t$$

- This equation holds for all t and ε_t
  Hence,

$$\begin{aligned} \alpha_0 &= 1 \\ \alpha_1 &= a_1 \alpha_0 \\ ... & \\ b_1 &= a_1 b_1 \end{aligned}$$

- We find that, if $a_1 \neq 1$, $b_1 = 0$ and $\alpha_i = a_1^i$. The particular solution is then:

$$y_t = \sum_{i=0}^{\infty} a^i \varepsilon_{t-i} \tag{28}$$

The same as (25)!

Lag Operators

- The lag operator L is defined to be a linear operator such that for any value y_t:

$$L^i y_t \equiv y_{t-i}$$

- There are a number of important properties regarding the Lag operator. Please see Hamilton or Enders.
- Example: Putting the Lag operator to work in our benchmark difference equation:

$$y_t = a_1 y_{t-1} + \varepsilon_t$$

Using the Lag operator $\Rightarrow$

$$y_t = a_1 L y_t + \varepsilon_t \tag{29}$$

Lag Operators

- The strength of the Lag operator is to make calculating the difference equation trivial, we can just solve (29) for y_t:

$$y_t = \frac{\varepsilon_t}{1 - a_1 L} \tag{30}$$

- For $|a| < 1$ we have the Lag operator property that $y_t/(1 - aL) = y_t(1 + aL + a^2L^2 + ...)$ Hence, (30) can be rewritten as

$$y_t = \sum_{i=0}^{\infty} a^i \varepsilon_{t-i} \tag{31}$$

Lag Operators

- The Lag Operator make it easy to extend the analysis to nth-order difference equations

$$y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + ... + a_n y_{t-n} + \varepsilon_t$$

$$(1 - a_1 L - a_2 L^2 - ... - a_n L^n) y_t = a_0 + \varepsilon_t$$

$$y_t = \frac{a_0 + \varepsilon_t}{(1 - a_1 L - a_2 L^2 - ... - a_n L^n)} \tag{32}$$

- The stability conditions can also be analysed with the Lag notation, but now the roots of the Lags must lie outside the unit root instead of inside.

Summary

- Time series were originally used for forecast
- In Macro they are particularly useful because they arise naturally in macro dynamic models
- We focused in this lecture on how to solve linear difference equations.
- Iteration can be useful but it is limited.
- The general solution to a difference equation can always be divided into two parts: a particular solution and all homogeneous solutions.
- The homogeneous solution is not unique.
- When we impose initial conditions, we can then have a unique solution.
- Finally we covered two methods that are useful for getting the particular solution.

Questions to think about

- Are difference equations able to capture economic behaviour in time series?
- What are convergent and divergent solutions?
- What is the relationship between stability conditions and convergence or divergence of a sequence?
- If we think about ε_t being stochastic instead of exogenous, what would be the statistical properties of a series? Do they differ if the series is convergent or divergent?